Solution of the generalized periodic discrete Toda equation

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Abstract

A box-ball system with more than one kind of balls is obtained by the generalized periodic discrete Toda equation (pd Toda eq.). We present an algebraic geometric study of the periodic Toda equation. The time evolution of pd Toda eq. is linearized on the Picard group of an algebraic variety, and theta function solutions are obtained.

1 Preface

A cellular automaton (CA) is a discrete dynamical system in which the dependent variables take on a finite set of discrete values. Although CAs evolve in time by simple evolution rules, they often show very complicated behaviour [1].

The box-ball system (BBS) is an important CA in which finitely many balls move in an array of boxes under a certain evolution rule [2, 3]. This discrete dynamical system is obtained from a discrete analogue of the Toda equation through the limiting procedure "ultradiscretization", and displays the behavioural characteristics of nonlinear integrable equations [4, 5]. In fact, the BBS has soliton-like solutions and a large number of conserved quantities [6, 7]. Moreover, the solution of the initial value problem of the periodic box-ball system (pBBS) with one kind of ball has been obtained by ultradiscretizing the solution of the periodic discrete Toda equation (pd Toda).

In 1999, Tokihiro, Nagai and Satsuma pointed out that the pBBS with M ($M \geq 1$) kind of balls (and with capacity one) is obtained by ultradiscretizing the generalized periodic discrete Toda equation [8]. Tokihiro and the author performed the ultradiscretization of theta function solutions for $M = 1$ of the pd Toda [9]. It is to be expected that the solution of the initial value problem of the pBBS with M kinds of balls can be obtained from the solution of the generalized Toda equation, as was the case for $M = 1$.

In this paper, we study the generalized pd Toda ($M \geq 1$):

$$I_n^{t+M} = I_n^t + V_n^t - V_{n-1}^{t+1}, \tag{1.1}$$

$$V_n^{t+1} = \frac{I_{n+1}^t V_n^t}{I_n^{t+M}}, \tag{1.2}$$

$$0 < \textstyle\prod_{n=1}^N V_n^t < \prod_{n=1}^N I_n^t, \tag{1.3}$$

$$0 < \textstyle\prod_{n=1}^N V_n^t < \prod_{n=1}^N I_n^{t+M-1}, \tag{1.4}$$

with periodicity condition $I_{n+N}^t = I_n^t, V_{n+N}^t = V_n^t$, where $N, n, t \in \mathbb{N}$.

The time evolution of this system is linearized on the Picard group of an algebraic variety. For example, the time evolution of pd Toda with $M = 1$ is linearized on Jacobi varieties of hyperelliptic curves [10]. For general M, the algebraic curves which appear in the linearization are more complicated than hyperelliptic curves.

These algebraic curves were analyzed by P. van Moerbeke and D. Mumford in 1978 [11]. In their work, they clarify the algebro-geometric properties of the curves which are defined by so-called *regular periodic operator* matrices. Although our matrices are not always regular operators, their results are essentially also applicable to our case.

The special feature of the discrete system (1.1-1.4) is that we can explicitly determine the action of the unit time evolution $t \mapsto t+1$ on the Picard group (proposition 2.16). Using this result, we extended the theta function expression of the solution of pd Toda to the $M \geq 1$ case.

The paper is organized as follows. In section 2, we introduce a *spectral curve* associated to the discrete system (1.1-1.4) and discuss its algebro-geometric properties. Our aim in this section is to determine the actions of the index shift $n \mapsto n+1$ and the time shift $t \mapsto t+1$ on a Picard group $\mathrm{Pic}^d(C)$ of this spectral curve C. In section 3, we give the theta function expression (theorem 3.2) which is the extension of the formula obtained by Kimijima and Tokihiro [10].

2 Spectral curve associated with pd Toda

2.1 The nature of the spectral curve

The pd Toda equation (1.1), (1.2) is equivalent to the following matrix form:

$$L_{t+1}(y) R_{t+M}(y) = R_t(y) L_t(y), \tag{2.1}$$

where $L_t(y)$ and $R_t(y)$ are given by

$$L_t(y) = \begin{pmatrix} 1 & & & V_N^t \cdot 1/y \\ V_1^t & 1 & & \\ & \ddots & \ddots & \vdots \\ & & V_{N-1}^t & 1 \end{pmatrix}, \quad R_t(y) = \begin{pmatrix} I_1^t & 1 & & \\ & I_2^t & \ddots & \\ & & \ddots & 1 \\ y & & & I_N^t \end{pmatrix},$$

and y is a complex variable. Let us introduce a new matrix $X_t(y)$ defined by

$$X_t(y) := L_t(y)R_{t+M-1}(y)\cdots R_{t+1}(y)R_t(y). \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$X_{t+1}(y)R_t(y) = R_t(y)X_t(y), \tag{2.3}$$

which implies that the eigenvalues of $X_t(y)$ are conserved quantities under the time evolution.

Lemma 2.1 *Assume that $\{I_n^t, V_n^t\}$ satisfies pd Toda equation (1.1–1.4). Then, $\prod_{n=1}^N V_n^t = \prod_{n=1}^N V_n^{t+1}$ and $\prod_{n=1}^N I_n^t = \prod_{n=1}^N I_n^{t+M}$.*

Proof. By (2.3), $\det X_t(y) = y^{-1}(y - \prod_n V_n^t)\,(\prod_n I_n^t - y)\cdots(\prod_n I_n^{t+M-1} - y)$ does not change under the time evolution for any y. Then, the set of quantities $\mathcal{U}_t := \{\prod_n V_n^t, \prod_n I_n^t, \ldots, \prod_n I_n^{t+M-1}\}$ satisfies $\mathcal{U}_t = \mathcal{U}_{t+1}$. From this equation, it follows that $\{\prod_n V_n^t, \prod_n I_n^t\} = \{\prod_n V_n^{t+1}, \prod_n I_n^{t+M}\}$. The inequality (1.3) and (1.4) leads the lemma. ■

Let $\widetilde{\Phi}(x, y) := \det(X_t(y) - xE)$ be the characteristic polynomial of $X_t(y)$ (E is the unit matrix). The equation

$$\widetilde{\Phi}(x, y) = 0 \tag{2.4}$$

defines the affine part $\widetilde{C}$ of its completion C. We call this projective curve C the *spectral curve* C of the pd Toda equation. C is a $(M+1)$-fold ramified covering over $\mathbb{P}^1$:

$$p_x : C \to \mathbb{P}^1, \tag{2.5}$$

and is also a N-fold ramified covering $p_y : C \to \mathbb{P}^1$. It goes without saying that C is conserved under the time evolution and is completely determined by the initial values $\{V_n^0, I_n^0, I_n^1, \ldots, I_n^{M-1}\}_{n=1}^N$. Note that C may fail to be smooth in certain situations. We restrict ourselves to the case where C is smooth.

Remark 2.1 *If $M < N$, the matrix $X_t(y)$ is a $N \times N$ matrix and is of the form*

$$X(y) = \begin{pmatrix} \alpha_1^{(1)} & \alpha_2^{(2)} & \cdots & \alpha_M^{(M)} & 1 & 0 & & & \\ \beta_1 & \alpha_2^{(1)} & \alpha_3^{(2)} & \cdots & \alpha_{M+1}^{(M)} & 1 & 0 & & \\ 0 & \beta_2 & \alpha_3^{(1)} & \alpha_4^{(2)} & \cdots & \alpha_{M+2}^{(M)} & 1 & 0 & \\ & 0 & \ddots & \ddots & \ddots & & \ddots & \ddots & \\ & & & & & & & & 1 \\ & & & & & & & & \alpha_N^{(M)} \\ & & & & & & \ddots & \ddots & \vdots \\ & & & & & & 0 & \beta_{N-1} & \alpha_N^{(1)} \end{pmatrix}$$

$$
+\begin{pmatrix} 0 & & & & & & \\ \vdots & & & & & & \\ 1 & & & & & & \\ \alpha_1^{(M)} & \ddots & & & & & \\ \vdots & \ddots & & & & & \\ \alpha_1^{(3)} & & \ddots & \ddots & & & \\ \alpha_1^{(2)} & \alpha_2^{(3)} & \cdots & \alpha_{M-1}^{(M)} & 1 & \cdots & 0 \end{pmatrix} \times y + \begin{pmatrix} & & \beta_N \\ & & \\ & & \end{pmatrix} \times \frac{1}{y}, \tag{2.6}
$$

where $\alpha_j^{(i)}, \beta_j, (1 \leq i \leq M, 1 \leq j \leq N)$ are polynomials in $\{V_n^t, I_n^t, I_n^{t+1}, \ldots, I_n^{t+M-1}\}_{n=1}^N$. In general, the (i,j)-component of $X(y)$ is the essentially finite summation $(X(y))_{i,j} = \sum_{l=-1}^{\infty} \alpha_j^{(j-i+lM+1)} y^l$, where $\alpha_j^{(-1)} = \beta_j$, $\alpha_j^{(M+1)} = 1$ and $\alpha_j^{(P)} = 0$ $(P < -1, P > M+1)$.

Remark 2.2 *When the greatest common divisor $(N, M) \neq 1$, this particular matrix (2.6) is not a regular periodic difference operator which is analyzed in [11].*

We now analyze the points contained in $C \setminus \widetilde{C}$. We calculate the polynomial expression of the function $\widetilde{\Phi}(x, y)$ in x, y and y^{-1}:

$$
\widetilde{\Phi}(x, y) = A_0(x) y^M + A_1(x) y^{M-1} + \cdots + A_M(x) + A_{M+1}(x) y^{-1} = 0. \tag{2.7}
$$

For the analysis of the behaviour of C, we need to analyze $A_j(x)$, $(j = 0, 1, \ldots, M+1)$. The following lemma is established by van Moerbeke and Mumford [11]. They analyzed the determinant $\det(X(y) - xE)$ by direct calculation.

Lemma 2.2 *Let $m := (N, M)$, $N = mN_1$ and $M = mM_1$. The polynomial $A_j(x)$ is a polynomial of degree k_j satisfying*

$$
k_j \leq \frac{jN}{M}, \ (0 \leq j \leq M), \quad k_{M+1} = 0. \tag{2.8}
$$

The equality in (2.8) holds if and only if the right hand side is an integer. Moreover, $A_0(x), A_{M_1}(x), A_{2M_1}(x), \ldots, A_{mM_1}(x)$ are expressed as:

$$
A_{rM_1}(x) = (-1)^{M(N-M)+r} \binom{m}{r} x^{rN_1} + \cdots, \quad (r = 0, 1, \ldots, m).
$$

We start form the polynomial (2.7). Let $\gamma := x^{N_1} y^{-M_1}$. By lemma 2.2, we obtain the expression

$$
\begin{aligned}
y^{-mM_1} \widetilde{\Phi}(x, y) &= (-1)^{M(N-M)} \left(\gamma^m - \tbinom{m}{1} \gamma^{m-1} + \tbinom{m}{2} \gamma^{m-2} + \cdots + (-1)^m \right) \\
&\qquad + \text{lower order terms when } |x|, |y| \to \infty \\
&\sim (-1)^{M(N-M)} (\gamma - 1)^m.
\end{aligned}
$$

This implies $x^{N_1}y^{-M_1} = \gamma \sim 1$ near $(x, y) = (\infty, \infty)$. By $(N_1, M_1) = 1$, there exists the local coordinate t equipped with the completion $C \supset \widetilde{C}$ such that

$$x \sim t^{-mM_1} = t^{-M},\ y \sim t^{-mN_1} = t^{-N}, \quad (x, y) \sim (\infty, \infty).$$

In particular, there exists *only one* point $P \in C$ which is expressed as $P = (\infty, \infty)$.

In a similar manner, there exists a point $Q \in C$ which is expressed as $Q = (\infty, 0)$. The local coordinate t near Q satisfies $x \sim t^{-1},\ y \sim t^{N}$. Using these fact, the divisors $(x), (y) \in \mathrm{Div}(C)$ are

$$(x) = -MP - Q + (\text{a positive divisor on } \widetilde{C}), \tag{2.9}$$
$$(y) = -NP + NQ. \tag{2.10}$$

Remark 2.3 *The existence of the unique point $P(\infty, \infty)$ is a special property of $X(y)$. In fact, there exist m points $P_j(x, y) = (\infty, \infty)$ $(j = 1, \ldots, m)$ on the algebraic curve associated to a regular operator matrix [11].*

Although the concrete calculations in [11] should be applied only to the case that $X(y)$ is a regular operator, these results are also applicable to our case on condition that C is smooth. Precisely, these calculations become true for our case by substituting

$$P_1 = P_2 = \cdots = P_m (= P). \tag{2.11}$$

2.2 The eigenvector mapping

We now define the *isolevel set* $\mathcal{T}_C$ as the set of matrices $X(y)$ associated with the the spectral curve C. The following proposition is fundamental to the algebro-geometric method for integrable systems.

Proposition 2.3 *Let C be smooth and $X(y) \in \mathcal{T}_C$. There is a unique line bundle $V \subset C \times \mathbb{C}^N$ s.t.*

$$\pi^{-1}(x, y) = \{\text{the eigenspace of } X(y) \text{ corresponding to the eigenvalue } x\} \subset \mathbb{C}^N,$$

for all $(x, y) \in C \setminus \{x = 0, \infty\}$ where $C \times \mathbb{C}^N \supset V \xrightarrow{\pi} C$ is called the canonical projection.

By virtue of this proposition, we obtain the map

$$\begin{array}{cccc} \varphi_C : & \mathcal{T}_C & \to & \{U \to \mathbb{P}^{N-1}\}_{U \subset C} \\ & X(y) & \mapsto & V^\vee \end{array},$$

where $V^\vee$ is the dual bundle of V. The section of $V^\vee$ is a component of the eigenvector of $X(y)$. By the Grothendieck-Riemann-Roch theorem, it follows that

$$\mathrm{Im}\,\varphi_C \subset \mathrm{Pic}^d(C), \quad d = g + N - 1, \tag{2.12}$$

where g is the genus of C. (See [12]).

Definition 2.1 *For smooth C, the eigenvector mapping associated to the equation* (*2.3*) *is the mapping*

$$\varphi_C : \mathcal{T}_C \to Pic^d(C)$$

defined as above. We shall call $V^\vee$ the eigenvector bundle.

The eigenvector mapping is an important tool to analyze the various integrable systems [13]. The following proposition is essential to our arguments in the present paper.

Proposition 2.4 *The eigenvector mapping $\varphi_C : \mathcal{T}_C \to Pic^d(C)$ is an isomorphism to* $\mathrm{Im}\varphi_C$.

This proposition is a straightforward result of the following theorem provided by van Moerbeke and Mumford.

Theorem 2.5 *There is a one-to-one correspondence between the two sets of data:*
(a)*a multi diagonal matrix $X(y)$ of the form (2.6) such that $\widetilde{\Phi}(x,y) = 0$ defines an affine part of a smooth curve.*
(b)*a smooth curve C, two points P, Q, two functions x, y on C and a divisor $\mathcal{D}$ which satisfies*

$$\varphi_C(X(y)) = \mathcal{D} + (N-1)Q. \tag{2.13}$$

C has genus $g = \dfrac{(N-1)(M+1)-m+1}{2}$, and $\deg \mathcal{D} = g$.

Remark 2.4 *Although the equation* (*2.13*) *does not appear in the van Moerbeke and Mumfords paper [11],we easily derive this equation from the relation* (*p.107*)

$$(g_k) + \mathcal{D} \geq \sum_{i=k+1}^{N} P_i - \sum_{i=k}^{N-1} Q_i, \quad k = 1, 2, \ldots, N-1, \tag{2.14}$$

where $(g_1, \ldots, g_{N-1}, 1)^T$ is a section of the eigenvector bundle $V^\vee$. In fact, (*2.14*) *yields*

$$(g_k)_\infty \leq \mathcal{D} + (N-k)Q, \tag{2.15}$$

which implies $d = \deg(\mathrm{Im}\varphi_C) \leq \deg \mathcal{D} + N - 1 = g + N - 1$. Because of the equality (*2.12*)*, $(g_1)_\infty$ must satisfy*

$$(g_1)_\infty = \mathcal{D} + (N-1)Q. \tag{2.16}$$

(*2.15*) *and* (*2.16*) *imply* (*2.13*).

Definition 2.2 *A finite component $\varphi_{\mathrm{fn}}(X(y))$ of the eigenvector mapping is a positive divisor $\mathcal{D}$ of degree g which appears in* (*2.13*).

2.3 The action of the evolutions on the eigenvector bundle

In this section, we represent the two actions — index evolution ($n \mapsto n+1$) and time evolution ($t \mapsto t+1$) — on the eigenvector bundle.

Proposition 2.6 *Let D_n be the divisor $D_n = P - Q$. Then the following diagram is commutative.*

$$\begin{array}{cccccc} & \mathcal{T}_C & \to & Pic^d(C) & \\ {\scriptstyle n\mapsto n+1} & \downarrow & & \downarrow & {\scriptstyle +D_n} \\ & \mathcal{T}_C & \to & Pic^d(C) & \end{array}$$

Proof. Let us denote $\sigma : n \mapsto n+1$. A straightforward calculation leads to

$$X(y)\begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \\ v_N \end{pmatrix} = x\begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \\ v_N \end{pmatrix} \Leftrightarrow \sigma^{-1}(X(y))\begin{pmatrix} y^{-1}v_N \\ v_1 \\ \vdots \\ v_{N-1} \end{pmatrix} = x\begin{pmatrix} y^{-1}v_N \\ v_1 \\ \vdots \\ v_{N-1} \end{pmatrix}. \tag{2.17}$$

It is enough to prove $(y^{-1}g_{N-1}^{-1})_\infty - (g_1)_\infty \sim -D_n$. By (2.14), we have $(y\, g_{N-1}) + \mathcal{D} \geq (N-1)Q - (N-1)P$. An argument similar to remark 2.4 allows to conclude $(y^{-1}g_{N-1}^{-1})_\infty \sim (y\, g_{N-1})_\infty = \mathcal{D} + (N-1)P$, which completes the proof. ■

In order to determine the action of the time evolution on the eigenvector bundle, we introduce the concepts of *Bloch solution* and *transposed operator*.

We identify the eigenvectors of $X(y)$ with the *Bloch solutions with multiplicity y* of the periodic infinite matrix

$$\widetilde{X} = \begin{pmatrix} \ddots & \ddots & & & \ddots & \ddots & & & & \\ \ddots & \alpha_N^{(1)} & \alpha_1^{(2)} & \cdots & \alpha_{M-1}^{(M)} & 1 & 0 & & & \\ & \beta_N & \alpha_1^{(1)} & \alpha_2^{(2)} & \cdots & \alpha_M^{(M)} & 1 & 0 & & \\ & 0 & \beta_1 & \alpha_2^{(1)} & \alpha_3^{(2)} & \cdots & \alpha_{M+1}^{(M)} & 1 & 0 & \\ & & 0 & \ddots & \ddots & & & \ddots & \ddots & \\ & & & & & & & & & \end{pmatrix},$$

which are the infinite vectors $\widetilde{\boldsymbol{v}} = (\cdots, v_{n-1}, v_n, v_{n+1}, \cdots)^T$ such that

$$\widetilde{X}\widetilde{\boldsymbol{v}} = x\widetilde{\boldsymbol{v}} \quad \text{and} \quad v_{n+N} = y\, v_n. \tag{2.18}$$

The first equation of (2.18) is equivalent to

$$\beta_{n-1}v_{n-1} + \alpha_n^{(1)}v_n + \alpha_{n+1}^{(2)}v_{n+1} + \cdots + \alpha_{n+M-1}^{(M)}v_{n+M-1} + v_{n+M} = xv_n \tag{2.19}$$

for $n \in \mathbb{Z}$ $\quad (\alpha_{n+N}^{(j)} = \alpha_n^{(j)}, \beta_{n+N} = \beta_n)$. Because the l.h.s. of (2.19) is a linear combination of $v_{n-1}, v_n, \cdots, v_{n+M-1}$, the Bloch solution associated with

$(x, y) \in C$ are also be written as a linear combination of $\widetilde{\boldsymbol{v}}^{(1)}, \widetilde{\boldsymbol{v}}^{(2)}, \ldots, \widetilde{\boldsymbol{v}}^{(M+1)}$, where

$$
\begin{aligned}
\widetilde{\boldsymbol{v}}^{(1)} &= (\ldots, 1, 0, \ldots, 0, v_{M+2}^{(1)}, v_{M+3}^{(1)}, \ldots)^T \\
\widetilde{\boldsymbol{v}}^{(2)} &= (\ldots, 0, 1, \ldots, 0, v_{M+2}^{(2)}, v_{M+3}^{(2)}, \ldots)^T \\
&\vdots \\
\widetilde{\boldsymbol{v}}^{(M+1)} &= (\ldots, 0, 0, \ldots, 1, v_{M+2}^{(M+1)}, v_{M+3}^{(M+1)}, \ldots)^T.
\end{aligned}
$$

More precisely, let $\psi(x, y)$ be the Bloch solution:

$$
\psi(x, y) = a_1 \widetilde{\boldsymbol{v}}^{(1)} + a_2 \widetilde{\boldsymbol{v}}^{(2)} + \cdots + a_{M+1} \widetilde{\boldsymbol{v}}^{(M+1)}, \tag{2.20}
$$

where $a_i = a_i(x, y)$ and $\widetilde{\boldsymbol{v}}^{(i)} = \widetilde{\boldsymbol{v}}^{(i)}(x)$.

Recalling (2.3) and proposition 2.3 the eigenvector $\boldsymbol{v}_t(x, y)$ at time t satisfies

$$
\boldsymbol{v}_{t+1}(x, y) = R_t(y) \cdot \boldsymbol{v}_t(x, y). \tag{2.21}
$$

Equivalently, the Bloch solution $\psi_t(x, y)$ satisfies

$$
\psi^{t+1}(x, y) = \widetilde{R}_t \cdot \psi^t(x, y), \tag{2.22}
$$

where

$$
\widetilde{R}_t = \begin{pmatrix}
\ddots & \ddots & & & & & \\
& I_N^t & 1 & & & & \\
& & I_1^t & 1 & & & \\
& & & I_2^t & \ddots & & \\
& & & & \ddots & 1 & \\
& & & & & I_N^t & \ddots \\
& & & & & & \ddots
\end{pmatrix}.
$$

(2.20) and (2.22) yield

$$
a_j^{t+1} = a_1^t (\widetilde{R}_t \widetilde{\boldsymbol{v}}^{(1)})_j + \cdots + a_{M+1}^t (\widetilde{R}_t \widetilde{\boldsymbol{v}}^{(M+1)})_j, \quad j = 1, 2, \ldots, M+1, \tag{2.23}
$$

where $(\boldsymbol{v})_j$ is a jth-component of the vector $\boldsymbol{v}$. Equation (2.23) is equivalent to

$$
\left(a_1^{t+1}, a_2^{t+1}, \ldots, a_M^{t+1}, a_{M+1}^{t+1}\right)^T = H_t \cdot \left(a_1^t, a_2^t, \ldots, a_M^t, a_{M+1}^t\right)^T,
$$

where

$$
H_t = \begin{pmatrix}
I_1^t & 1 & & & \\
& I_2^t & \ddots & & \\
& & \ddots & 1 & \\
& & & I_M^t & 1 \\
v_{M+2}^{(1)} & v_{M+2}^{(2)} & \cdots & v_{M+2}^{(M)} & I_{M+1}^t + v_{M+2}^{(M+1)}
\end{pmatrix}. \tag{2.24}
$$

Using this equation, we obtain

$$\left(v_1^{t+1}, v_2^{t+1}, \dots, v_M^{t+1}, v_{M+1}^{t+1}\right)^T = H_t \cdot \left(v_1^t, v_2^t, \dots, v_M^t, v_{M+1}^t\right)^T, \tag{2.25}$$

where $\psi^t(x,y) = (\dots, v_1^t, v_2^t, \dots, v_{M+1}^t, \dots)^T$.

Lemma 2.7 *For fixed generic x, the $M+1$ Bloch solutions associated with x are linearly independent.*

Proof. For generic x, associated multiplicities y_j $(j = 1, 2, \dots, M+1)$ of the Bloch solutions are all distinct. ■

Lemma 2.8 $\det H_t = (-1)^{M+1} I_1^t x$.

This lemma is proved by an elementary calculation, which we shall give in the appendix.

Let $\mathcal{F}$ be the invertible sheaf associated with the eigenvector bundle $V^\vee$. Let us consider the direct image $(p_x)_*\mathcal{F}$, where p_x is a projection of C defined by (2.5). The sheaf $(p_x)_*\mathcal{F}$ is a locally free sheaf of rank $M+1$. By lemma 2.7, the fiber of this direct sheaf is $\mathbb{C}^{M+1}$.

Equation (2.25) is regarded as the time action to the space $\mathbb{C}^{M+1}$. Recall that the components of the Bloch solutions of $\widetilde{X}$ are the section of the sheaf $\mathcal{F}$. For $x \in \mathbb{P}^1$, we denote the k-th component of the infinite vector $\psi^t(x, y_j)$ by $v_{jk}^t(x)$, and $\psi^t(x, y_j)$ by $\psi_j^t(x)$. By (2.20), the finite vectors $\widehat{\psi}_j^t := (v_{j,1}^t, v_{j,2}^t, \dots, v_{j,M+1}^t)^T$ have the property

$$\{\psi_j^t\}_{j=1}^{M+1} \text{ are linearly independent} \iff \{\widehat{\psi}_j^t\}_{j=1}^{M+1} \text{ are linearly independent.} \tag{2.26}$$

Equation (2.25) implies that

$$(\widehat{\psi}_1^{t+1}, \widehat{\psi}_2^{t+1}, \dots, \widehat{\psi}_{M+1}^{t+1}) = H_t \cdot (\widehat{\psi}_1^t, \widehat{\psi}_2^t, \dots, \widehat{\psi}_{M+1}^t). \tag{2.27}$$

On the other hand, lemma 2.7, 2.8 and (2.26) imply that there exists at least one vector $\widehat{\psi}_j^t$ which satisfies

$$\mathrm{mult}_x \widehat{\psi}_j^{t+1} > \mathrm{mult}_x \widehat{\psi}_j^t, \tag{2.28}$$

where $\mathrm{mult}_x(z_1, z_2, \dots, z_N)^T = \min[\, \mathrm{mult}_x z_1, \mathrm{mult}_x z_2, \dots, \mathrm{mult}_x z_N \,]$ and $\mathrm{mult}_x z$ is multiplicity of x in z.

The same discussion can be repeated for the projection

$$p_y : C \to \mathbb{P}^1,$$

which is an N-fold ramified covering over $\mathbb{P}^1$.

The following two facts are then obvious to prove.

Lemma 2.9 *For fixed generic y, the N eigenvectors of $X_t(y)$ are linearly independent.*

Lemma 2.10 $\det R_t(y)$ *is a polynomial of degree one in* y.

Let us consider the direct image $(p_y)_*\mathcal{F}$. From lemma 2.9 we then find that the fiber of this direct image is $\mathbb{C}^N$.

Equation (2.21) can be regarded as the time action to the space $\mathbb{C}^N$. For fixed y, we denote $p_y^{-1}(x) = \{(x_1, y), \ldots, (x_N, y)\}$. Then we obtain that

$$\exists j \quad \text{s.t.} \quad \mathrm{mult}_{(y-y_t)} \boldsymbol{v}_{t+1}(x_j, y) > \mathrm{mult}_{(y-y_t)} \boldsymbol{v}_t(x_j, y), \tag{2.29}$$

where $\det R_t(y_t) = 0$ ($\Leftrightarrow y_t = \prod_n I_n^t$).

2.4 The transposed operator

In this subsection, we introduce the *transposed operator*, and give the proof of proposition 2.16.

The section of the eigenvector bundle $V^\vee$ can be described as a rational function of x and y. We denote the set of these sections by $\Gamma(V^\vee)$. Let $\Delta_{i,j} := (-1)^{i+j} \times (i,j)$-th minor of $X(y) - xE$. We have $g_k = \frac{\Delta_{N,k}}{\Delta_{N,N}}$ $(k = 1, 2, \ldots, N-1)$, where $(g_1, \ldots, g_{N-1}, 1)^T \in \Gamma(V^\vee)$. We are interested in the divisor $(g_1)_\infty$ and hence, it is important to explore the common zeros of $\Delta_{N,1}$ and $\Delta_{N,N}$.

Equation (2.3) is equivalent to

$$X_t(y)^T R_t(y)^T = R_t(y)^T X_{t+1}(y)^T. \tag{2.30}$$

Let us define $s := -t$, and $A^\star := J A^T J$ for a regular matrix A where

$$J = \begin{pmatrix} 0 & & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix}.$$

Then (2.30) becomes

$$X^\star_{s+1} R^\star_{s+1} = R^\star_{s+1} X^\star_s. \tag{2.31}$$

Note that the matrix $X^\star(y)$ is also of the form (2.6). We call this new matrix $X^\star$ the *transposed operator* of X. A careful analysis of the eigenvector bundle of the transposed operator gives more information on the original matrix X.

Lemma 2.11 (van Moerbeke, Mumford) (1) *There exist positive regular divisors* $\mathcal{D}_1$, $\mathcal{D}_2$, *and* $\mathcal{D}_3$ *such that*

$$(\Delta_{N,1}/\Delta_{N,N}) = \mathcal{D}_1 + (N-1)P - \varphi_{\mathrm{fn}}(X) - (N-1)Q, \tag{2.32}$$

and

$$(\Delta_{1,N}/\Delta_{N,N}) = \mathcal{D}_2 + (N-1)Q - \mathcal{D}_3 - (N-1)P. \tag{2.33}$$

Moreover, $\deg \mathcal{D}_i = g$ $(i = 1, 2, 3)$.
(2) *The divisor* $(\Delta_{N,N})$ *satisfies*

$$(\Delta_{N,N}) = \varphi_{\mathrm{fn}}(X) + \mathcal{D}_3 - (NM - M - n + 1)P - NQ. \tag{2.34}$$

Remark 2.5 *Lemma 2.11* (1) *and Remark 2.16 imply that* $\deg \frac{\Delta_{1,N}}{\Delta_{N,N}} = g+N-1$, *and that* $\mathcal{D}_2$ *and* $\mathcal{D}_3$ *do not have common points.*

Lemma 2.12 $\varphi_{\rm fn}(X^\star) = \varphi_{\rm fn}(X^T) = \mathcal{D}_2$.

Proof. Note that $X^\star$ and X^T give the same spectral curve C. By definition, it follows that: $X^T(J\boldsymbol{v}) = x(J\boldsymbol{v}) \Leftrightarrow X^\star\boldsymbol{v} = x\boldsymbol{v}$, which implies the first equality of the lemma.
The second equality is obtained from (2.13), (2.33) and the fact that the $(N,1)$-th minor of X is the $(1,N)$-th minor of X^T. ■

Lemma 2.12 and (2.33) yield

$$\mathcal{D}_3 \sim \varphi_{\rm fn}(X^\star) + (N-1)Q - (N-1)P \sim \varphi_{\rm fn}(X^\star) - Q + P.$$

Using proposition 2.6, we obtain $\mathcal{D}_3 \sim \varphi_{\rm fn}(\sigma(X^\star)) = \varphi_{\rm fn}((\sigma^{-1}X)^\star)$. In fact, by virtue of the Riemann-Roch theorem, we obtain the following stronger result:

$$\mathcal{D}_3 = \varphi_{\rm fn}((\sigma^{-1}X)^\star)$$

because $\mathcal{D}_3$ is a regular divisor of degree g.

Proposition 2.13 $\Delta_{N,N}$ *has* $2g$ *zeros in* $C \setminus \{x=\infty\}$. *And the corresponding divisor is*

$$\varphi_{\rm fn}(X) + \varphi_{\rm fn}((\sigma^{-1}X)^\star).$$

As a corollary of this proposition, the zeros of $\Delta_{1,1}, \Delta_{1,N}, \Delta_{N,1}$ are also determined:

Corollary 2.14 $\Delta_{1,1}, \Delta_{1,N}, \Delta_{N,1}$ *have* $2g$ *zeros in* $C \setminus \{x=\infty\}$ *respectively. The corresponding divisors are:*

$$\begin{array}{lcrcl} \Delta_{1,1} & : & \varphi_{\rm fn}(\sigma^{-1}X) & + & \varphi_{\rm fn}(X^\star) \\ \Delta_{1,N} & : & \varphi_{\rm fn}(X) & + & \varphi_{\rm fn}(X^\star) \\ \Delta_{N,1} & : & \varphi_{\rm fn}(\sigma^{-1}X) & + & \varphi_{\rm fn}((\sigma^{-1}X)^\star). \end{array} \tag{2.35}$$

Proof. The divisor of $\Delta_{1,1}$ is obtained from $\Delta_{N,N}$ by the change of the index: $n \to n-1$. The remaining divisors are determined by the identity $\Delta_{1,1}\Delta_{N,N} = \Delta_{1,N}\Delta_{N,1}$ and (2.16). ■

Example $(N=4, M=2)$

$$X = \begin{pmatrix} \alpha_1^{(1)} & \alpha_2^{(2)} & 1 & \beta_4 1/y \\ \beta_1 & \alpha_2^{(1)} & \alpha_3^{(2)} & 1 \\ y & \beta_2 & \alpha_3^{(1)} & \alpha_4^{(2)} \\ \alpha_1^{(2)} y & y & \beta_3 & \alpha_4^{(1)} \end{pmatrix}, X^\star = \begin{pmatrix} \alpha_4^{(1)} & \alpha_4^{(2)} & 1 & \beta_4 1/y \\ \beta_3 & \alpha_3^{(1)} & \alpha_3^{(2)} & 1 \\ y & \beta_2 & \alpha_2^{(1)} & \alpha_2^{(2)} \\ \alpha_1^{(2)} y & y & \beta_1 & \alpha_1^{(1)} \end{pmatrix},$$

$$\sigma^{-1}X = \begin{pmatrix} \alpha_4^{(1)} & \alpha_1^{(2)} & 1 & \beta_3 1/y \\ \beta_4 & \alpha_1^{(1)} & \alpha_2^{(2)} & 1 \\ y & \beta_1 & \alpha_2^{(1)} & \alpha_3^{(2)} \\ \alpha_4^{(2)} y & y & \beta_2 & \alpha_3^{(1)} \end{pmatrix}, (\sigma^{-1}X)^\star = \begin{pmatrix} \alpha_3^{(1)} & \alpha_3^{(2)} & 1 & \beta_3 1/y \\ \beta_2 & \alpha_2^{(1)} & \alpha_2^{(2)} & 1 \\ y & \beta_1 & \alpha_1^{(1)} & \alpha_1^{(2)} \\ \alpha_4^{(2)} y & y & \beta_4 & \alpha_4^{(1)} \end{pmatrix}.$$

The correspondence $X \leftrightarrow (\sigma^{-1}X)^\star$ is equivalent to the correspondence:

$$\alpha_i^{(1)} \leftrightarrow \alpha_{4-i}^{(1)}, \quad \alpha_i^{(2)} \leftrightarrow \alpha_{1-i}^{(2)}, \quad \beta_i \leftrightarrow \beta_{3-i}.$$

Our aim is to analyze the distribution of common zeros of $\Delta_{N,1}$ and $\Delta_{N,N}$. This set of zeros has the following characteristic property.

Lemma 2.15

$$\begin{aligned}&\{\textit{The common zeros of } \{\Delta_{N,1}, \Delta_{N,N}\}\} \\ &= \{\textit{The common zeros of } \{\Delta_{N,k}\}_{k=1,2,\dots,N}\}.\end{aligned}$$

Proof. Consider the following open subset on C: $\widetilde{C} = C \setminus \{x = \infty\}$. By (2.15) and (2.16), for $k \geq 2$, we obtain $(g_k)_\infty|_{\widetilde{C}} < (g_1)_\infty|_{\widetilde{C}}$. This is equivalent to $(\Delta_{N,k}/\Delta_{N,N})_\infty < (\Delta_{N,1}/\Delta_{N,N})_\infty$, which proves the lemma. ■

Using the preceding calculations, we come to the linearization result.

Proposition 2.16 *Let $D_{(j)}$ $(j = 1, 2, \dots, M)$ be the divisors $D_{(j)} = A_j - Q$, where $A_j = (0, y_j)$ and y_j is the complex number which satisfies $y_j = \prod_n I_n^j$. If $t \equiv j \pmod M$, the following diagram is commutative.*

$$\begin{array}{cccc} & \mathcal{T}_C & \rightarrow & Pic^d(C) \\ {\scriptstyle t\mapsto t+1} & \downarrow & & \downarrow \quad {\scriptstyle +D_{(j)}} \\ & \mathcal{T}_C & \rightarrow & Pic^d(C) \end{array}$$

Proof. Recall that $\prod_n I_n^t = \prod_n I_n^{t+M}$ by lemma 2.1. Let $t \equiv j \pmod M$. Proposition 2.13 and corollary 2.14 imply

$$\{\text{Common zeros of } \Delta_{N,1} \text{ and } \Delta_{N,N}\} = \varphi_{\text{fn}}((\sigma^{-1}X)^\star).$$

Using this fact, it becomes clear that the equations (2.28) and (2.29) give the time evolution action of the transposed operator $X^\star$. In fact, by virtue of lemma 2.15, (2.28) requires the existence of a point $(x, y) = (0, y')$ which becomes the common zero of $\Delta_{N,1}$ and $\Delta_{N,N}$ when $t \mapsto t + 1$. Then (2.29) also requires the existence of a point $(x, y) = (x', y_j)$ which becomes the common zero of $\Delta_{N,1}$ and $\Delta_{N,N}$. Using the same argument about poles, it follows that the divisor $\mathcal{D}'_{(j)} = Q - A_j$ obeys the relation: $\varphi_{\text{fn}}(X^\star_{t+1}) = \varphi_{\text{fn}}(X^\star_t) + \mathcal{D}'_{(j)}$. Recalling (2.31), the two time evolutions defined by X_t and $X^\star_s$ are opposite: $s = -t$. Hence the divisor $\mathcal{D}_{(j)} = -\mathcal{D}'_{(j)} = A_j - Q$ gives the time evolution of original system. ■

3 Theta function solutions

3.1 The distribution of the points $\varphi_{\text{fn}}(X)$

Let $\varphi_{\text{fn}}(X) = P_1 + \cdots + P_g$, $P_j = (x_j, y_j) \in C$. We are interested in the numbers $x_j \in \mathbb{C}$, $(j = 1, 2, \dots, g)$. Now, consider the resultant of the polynomials [14] $\Phi(x,y) = y \det(X(y) - xE)$ and $\Delta_{N,N}$ as polynomials in y. Let us denote this

resultant by $\mathrm{Rest}_y(\Phi, \Delta_{N,N}) =: R(x)$, which is a polynomial in x. More precisely, $R(x)$ is an element of $\mathbb{C}[\alpha_1^{(1)}, \ldots, \alpha_N^{(1)}; \ldots; \alpha_1^{(M)}, \ldots, \alpha_N^{(M)}; \beta_1, \ldots, \beta_N][x]$. From proposition 2.13, it follows that $\deg_x R(x) = 2g$.

We also consider the resultant $S(x) := \mathrm{Rest}_y(\Phi, \Delta_{1,N})$. Due to proposition 2.13 and corollary 2.14 the common divisor of $R(x)$ and $S(x)$ is an element of $\mathbb{C}(\alpha_1^{(1)}, \ldots, \alpha_N^{(1)}; \ldots; \alpha_1^{(M)}, \ldots, \alpha_N^{(M)}; \beta_1, \ldots, \beta_N)[x]$, the degree of which (as a polynomial in x) is equal to g. Multiplying divisors (if needed), we obtain the monic polynomial

$$\Upsilon(x) \in \mathbb{C}(\alpha_1^{(1)}, \ldots, \alpha_N^{(1)}; \ldots; \alpha_1^{(M)}, \ldots, \alpha_N^{(M)}; \beta_1, \ldots, \beta_N)[x],$$

the roots of which are the common roots of $R(x)$ and $S(x)$, i.e., the common roots of $\Delta_{1,N}$ and $\Delta_{N,N}$. Recalling that the set of the common zeros of $\Delta_{N,N}$ and $\Delta_{1,N}$ contained in $C \setminus \{x = \infty\}$ is $\{P_1, \ldots, P_g\}$, we conclude:

$$\deg_x \Upsilon(x) = g, \quad \Upsilon(x_i) = 0, \quad i = 1, 2, \ldots, g. \tag{3.1}$$

3.2 Theta function solutions

For a complex curve (or Riemann surface) C of genus g, one usually considers a *canonical basis of* $H_1(C, \mathbb{Z})$. We denote the canonical basis by $a_1, \ldots, a_g$; $b_1, \ldots, b_g \in H_1(C, \mathbb{Z})$. Let $\omega_1, \ldots, \omega_g$ be the holomorphic differential of C which satisfies $\int_{a_j} \omega_i = \delta_{j,i}$. A *period matrix* of the Riemann surface C is a $g \times g$ matrix $B = (\int_{b_j} \omega_i)$. Let $\theta(\boldsymbol{z}, B)$ be the *theta function*; $\mathbb{C}^g \to \mathbb{C}$, and $\boldsymbol{A}$: $\mathrm{Pic}^g \xrightarrow{\sim} J(C)\,(:= \mathbb{C}^g/(\mathbb{Z}^g + B\mathbb{Z}^g))$ the *Abelian mapping*. The following theorem is a classical and fundamental result.

Theorem 3.1 (Riemann) *Let C be a Riemann surface of genus g, and let $\mathcal{D} = P_1 + \cdots + P_g$ be a regular positive divisor. Then the function*

$$F(p) = \theta(\boldsymbol{A}(p) - \boldsymbol{A}(\mathcal{D}) - \boldsymbol{K}, B), \quad p \in C$$

has exactly g zeros $p = P_1, \ldots, P_g$ on C, where $\boldsymbol{K}$ is the Riemann constant of C.

To obtain the solution to the pd Toda equation, we consider the following integral:

$$I = \frac{1}{2\pi i} \int_{\partial C^\circ} x(p) \frac{dF(p)}{F(p)} \left(=: \frac{1}{2\pi i} \int_{\partial C^\circ} d\mathcal{Z}(p) \right). \tag{3.2}$$

Here the integral path ∂C° goes along the edge of the simply connected domain C° obtained from the Riemann surface by cutting it along $a_1, \ldots, a_g; b_1, \ldots, b_g$.

The integral I can be rewritten as

$$\begin{aligned} I &= \tfrac{1}{2\pi i} \textstyle\sum_{k=1}^g \left(\int_{a_k} + \int_{a_k^{-1}} + \int_{b_k} + \int_{b_k^{-1}} \right) d\mathcal{Z}(p) \\ &= \tfrac{1}{2\pi i} \textstyle\sum_{k=1}^g \left(\int_{a_k} \{d\mathcal{Z}(p) - d\mathcal{Z}(p + b_k)\} + \int_{b_k} \{d\mathcal{Z}(p) - d\mathcal{Z}(p + a_k)\} \right). \end{aligned}$$

Recalling the classical fact $F(p+a_k) = F(p), F(p+b_k) = \exp(-2\pi i(\boldsymbol{A}(p) - \boldsymbol{A}(\mathcal{D}) - \boldsymbol{K})_k)F(p)$ and $d(\boldsymbol{A}(p))_k = \omega_k(p)$, the integral I is transformed to

$$I = \textstyle\sum_{k=1}^{g} \int_{a_k} x(p)\,\omega_k(p). \tag{3.3}$$

On the other hand, by the residue theorem, the integral I also has the expression:

$$I = \textstyle\sum_{i=1}^{g} x(P_i) + \mathrm{Res}_P(d\mathcal{Z}) + \mathrm{Res}_Q(d\mathcal{Z}). \tag{3.4}$$

Let t_P and t_Q be local coordinates around P and Q respectively. These satisfy

$$x \sim 1/(t_P)^M, \text{ (neighbor of } P), \quad x \sim 1/t_Q, \text{ (neighbor of } Q).$$

In a neighborhood of P, one has:

$$d\mathcal{Z} \sim \frac{1}{(t_P)^M}\frac{d\log F}{dt_P}\,dt_P = \frac{1}{(t_P)^M}\sum_{l=1}^{g}(\partial_l \log F)\left(\frac{d(\boldsymbol{A})_l}{dt_P}\right)dt_P \tag{3.5}$$

To calculate the residue of the differential (3.5), we explore the behaviour of $\frac{d(\boldsymbol{A})_l}{dt_P}dt_P = \omega_l$ around points P and Q. Let $c_l := \mathrm{Res}_P(\omega_l/(t_P)^M)$. Then we obtain the expression:

$$\mathrm{Res}_P(d\mathcal{Z}) = \textstyle\sum_{l=1}^{g} c_l(\partial_l \log(F(P))). \tag{3.6}$$

In the similar manner, we also conclude

$$\mathrm{Res}_Q(d\mathcal{Z}) = \textstyle\sum_{l=1}^{g} c'_l(\partial_l \log(F(Q))), \tag{3.7}$$

where $c'_l := \mathrm{Res}_Q(\omega_l/t_Q)$.

By (3.3) and (3.4), we obtain

$$\sum_{l=1}^{g} x(P_l) = \sum_{l=1}^{g}\int_{a_l} x(p)\,\omega_l(p) - \sum_{l=1}^{g} c_l(\partial_l \log(F(P))) - \sum_{l=1}^{g} c'_l(\partial_l \log(F(Q))). \tag{3.8}$$

Using this equation, we obtain the following theorem which is a generalization of the preceding result concerning the theta function solution to the pd Toda ($M=1$) equation [10] and which is the main theorem in the present paper:

Theorem 3.2 *Let $\Upsilon(x)$ be the monic polynomial of degree g obtained by (3.1):*

$$\Upsilon(x) = x^g - a_1 x^{g-1} + \cdots + (-1)^g a_g,$$

with $a_1, \ldots, a_g \in \mathbb{C}(\alpha_1^{(1)}, \ldots, \alpha_N^{(1)}; \ldots; \alpha_1^{(M)}, \ldots, \alpha_N^{(M)}; \beta_1, \ldots, \beta_N)$. Then we find

$$\begin{aligned} a_1 \quad &= \textstyle\sum_{l=1}^{g}\int_{a_l} x(p)\,\omega_l(p) - \sum_{l=1}^{g} c_l\,\{\partial_l \log\theta(n\boldsymbol{k} + \boldsymbol{\nu}(t) + \boldsymbol{c}_0, B)\} \\ &\quad - \textstyle\sum_{l=1}^{g} c'_l\,\{\partial_l \log\theta((n+1)\boldsymbol{k} + \boldsymbol{\nu}(t) + \boldsymbol{c}_0, B)\}, \end{aligned} \tag{3.9}$$

where $c_l = Res_P(\omega_l/(t_P)^M)$, $c'_l = Res_Q(\omega_l/t_Q)$, $\boldsymbol{k} = \boldsymbol{A}(P-Q)$, $\boldsymbol{c}_0 = \boldsymbol{A}(Q - \mathcal{D}_0 + \Theta)$, and $\boldsymbol{\nu}(pM+q) = p\boldsymbol{A}(A_1 + \cdots + A_M) + \boldsymbol{A}(A_1 + \cdots + A_q) - (pM+q)\boldsymbol{A}(Q)$ $(1 \le q \le M)$. Here Θ is the theta divisor: $\boldsymbol{A}(\Theta) = -\boldsymbol{K}$. The divisor $\mathcal{D}_0$ is the initial value $\varphi_{\mathrm{fn}}(X_{t=0}(y)) = \mathcal{D}_0$.

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A Proof of lemma 2.8

In this section, we give the proof of lemma 2.8. Let

$$U_k := R_{t+k-1}(y)R_{t+k-2}(y)\dots R_t(y), \quad 1 \leq k \leq M.$$

Define the row vector $\boldsymbol{u}^{(k)} = (u^{(k)}_{1,1}, u^{(k)}_{1,2}, \dots, u^{(k)}_{1,N})$, where $U_k = (u^{(k)}_{i,j})$, and the following homomorphism of rings:

$$\sigma : \mathbb{Z}\left[\{V^t_n, I^t_n\}_{n\in\mathbb{Z}}\right] \to \mathbb{Z}\left[\{V^t_n, I^t_n\}_{n\in\mathbb{Z}}\right] \quad ; \quad V^t_n \mapsto V^t_{n+1},\, I^t_n \mapsto I^t_{n+1}.$$

We rewrite the j-th component of $\boldsymbol{u}^{(k)}$ as $u^{(k)}_j$ for short. By definition of $R_t(y)$, we obtain

$$u^{(k+1)}_j = \sigma(u^{(k)}_{j-1}) + I^{t+k}_1 u^{(k)}_j, \quad (u^{(k)}_{-1} = 0). \tag{A.1}$$

On the other hand, the 2nd row of the matrix $X_t(y)$, which is of the form

$$(\beta_1, \alpha^{(1)}_2, \alpha^{(2)}_3, \dots, \alpha^M_{M+1}, 1, 0, \dots, 0),$$

satisfies $\beta_1 = V^t_1 u^{(M)}_1$, $\alpha^{(j)}_{j+1} = \sigma(u^{(M)}_j) + V^t_1 u^{(M)}_{j+1}$. Figure 1 displays the algorithm which we shall use to obtain the row vector $\boldsymbol{u}^{(k)}$ expressed by (A.1).

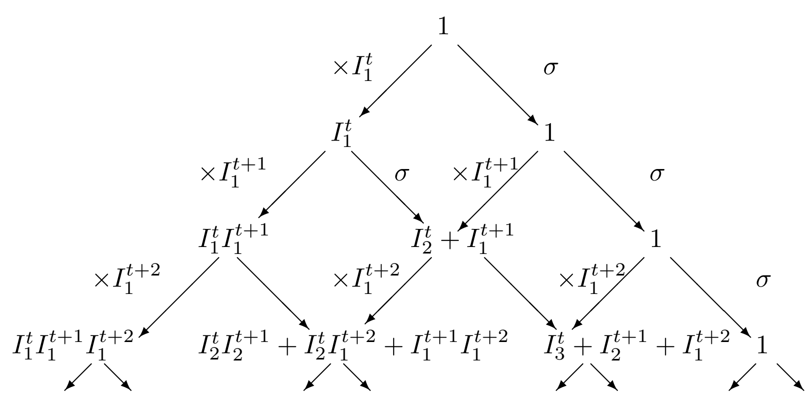

Figure 1: The $j+1$-th row of this diagram displays the first $j+1$ non-zero components of the row vector $\boldsymbol{u}^{(j)}$.

Let us introduce the signs $\swarrow$ and $\searrow$ to describe the terms which appear in the figure 1. Let us define the set of arrows $Ar := \{\swarrow, \searrow\}$ and the set of sequences $Ar^r := \{(a_1, a_2, \dots, a_r) \,|\, a_j \in Ar\}$, $r \in \mathbb{N}$. We define the map of sets $\{\cdot\} : \bigcup_r Ar^r \to \mathbb{Z}[\{I^{t+l}_n\}_{1\leq n\leq N, 1\leq l\leq M}]$ as follows:

For $r=0$, define $\{\emptyset\} := 1$. If $\{a_1, a_2, \ldots, a_r\} \in \mathbb{Z}[\{I_n^{t+l}\}_{n,l}]$ is given, we define $\{a_1, a_2, \ldots, a_r, a_{r+1}\}$ by

$$\{a_1, \ldots, a_r, \swarrow\} := I_1^{t+r}\{a_1, a_2, \ldots, a_r\}, \quad \{a_1, \ldots, a_r, \searrow\} := \sigma(\{a_1, \ldots, a_r\})$$

inductively. For example, $\{\swarrow\} = I_1^t$, $\{\swarrow\swarrow\} = I_1^t I_1^{t+1}$, $\{\swarrow\searrow\} = I_2^t$, $\{\searrow\swarrow\} = I_1^{t+1}$. By definition, the j-th component of $\boldsymbol{u}^{(k)}$ satisfies

$$u_j^{(k)} = \sum_{\sharp\searrow = j-1,\ \sharp\swarrow = k-j+1} \{a_1, \ldots, a_k\}. \tag{A.2}$$

Lemma A.1 $\{\swarrow, a_2, \ldots, a_k\} = I_{l+1}^t\{\searrow, a_2 \ldots, a_k\}$,
where $l = \sharp\{$*the arrow* $\searrow$ *included in* $\{a_2, \ldots, a_k\}\}$.

Proof. We prove the equation by induction respected to k. If $k=1$, the equation is equivalent to $\{\swarrow\} = I_1^t\{\searrow\}$ which is true by definition. Let $k \geq 2$. If $a_k = \swarrow$, we obtain l.h.s $= I_1^{t+k-1}\{\swarrow, a_2, \ldots, a_{k-1}\}$, r.h.s $= I_1^{t+k-1} I_{l+1}^t \{\searrow, a_2, \ldots, a_{k-1}\}$. By assumption of induction, it follows that l.h.s = r.h.s. We also prove the equation in the similar manner if $a_k = \searrow$. ■

Lemma A.2 $u_1^{(M)} + \sum_{j=1}^M (-1)^j u_{j+1}^{(M)} I_1^t \cdots I_j^t = 0$.

Proof.

$$\begin{aligned}
& u_1^{(M)} + \textstyle\sum_{j=1}^M (-1)^j u_{j+1}^{(M)} I_1^t \cdots I_j^t \\
& = \{\swarrow\swarrow \cdots \swarrow\} + \sum_{j=1}^M (-1)^j \left[\sum_{\sharp\searrow = j,\ \sharp\swarrow = M-j} \{\swarrow, *, \ldots, *\} + \{\searrow, *, \ldots, *\} \right] I_1^t \cdots I_j^t \\
& = \{\swarrow\swarrow \cdots \swarrow\} + \sum_{j=1}^{M-1} (-1)^j \sum_{\sharp\searrow = j+1, \sharp\swarrow = M-j-1} \{\searrow, *\}\, I_1^t \cdots I_j^t I_{j+1}^t \\
& + \sum_{j=1}^M (-1)^j \sum_{\sharp\searrow = j, \sharp\swarrow = M-j} \{\searrow, *\}\, I_1^t \cdots I_j^t \qquad (\because \text{Lemma A.1}) \\
& = \{\swarrow\swarrow \cdots \swarrow\} - I_1\{\searrow\swarrow \cdots \swarrow\} \\
& = 0 \quad ■
\end{aligned}$$

Lemma A.3 $\sum_{j=1}^M (-1)^j \sigma(u_j^{(M)}) I_1^t \cdots I_j^t = (-1)^M I_1^t \cdots I_{M+1}^t$.

Proof. We start from lemma A.2.

$$\begin{aligned}
0 &= \sigma\left(u_1^{(M)} + \textstyle\sum_{j=1}^M (-1)^j u_{j+1}^{(M)} I_1^t \cdots I_j^t\right) \\
&= \sigma(u_1^{(M)}) + \textstyle\sum_{j=1}^M (-1)^j \sigma(u_{j+1}^{(M)}) I_2^t \cdots I_{j+1}^t.
\end{aligned}$$

Multiplying I_1, we obtain

$$\begin{aligned} 0 &= \textstyle\sum_{j=1}^{M+1}(-1)^j\sigma(u_j^{(M)})I_1^t\cdots I_j^t \\ &= \textstyle\sum_{j=1}^{M}(-1)^j\sigma(u_j^{(M)})I_1^t\cdots I_j^t + (-1)^{M+1}I_1^t\cdots I_{M+1}^t, \end{aligned}$$

which complete the proof. ■

proof of lemma 2.8

Calculate $\det H_t$ by the definition of H_t (2.24). The components $v_{M+2}^{(j)}$, $(j = 1, 2, \ldots, M+1)$ are rewritten as

$$v_{M+2}^{(1)} = -\beta_1,\ v_{M+2}^{(2)} = x - \alpha_2^{(1)},\ v_{M+2}^{(j)} = -\alpha_j^{(j-1)},\ (j = 3, 4, \ldots, M+1).$$

by virtue of (2.19). The expansion of the determinant with respect to the $(M+1)$-st row yields

$$\begin{aligned} \det H_t &= (-1)^M\{-\beta_1 + (\alpha_2^{(1)} - x)I_1^t - \alpha_3^{(2)}I_1^tI_2^t + \cdots \\ &\qquad + (-1)^M(I_{M+1}^t - \alpha_{M+1}^{(M)})I_1^t\cdots I_M^t\} \\ &= (-1)^M\left\{-\beta_1 - xI_1^t - \textstyle\sum_{j=1}^{M}(-1)^j\alpha_{j+1}^{(j)}I_1^t\cdots I_j^t + (-1)^M I_1^t\cdots I_M^t I_{M+1}^t\right\} \\ &= (-1)^M\Big\{-V_1^tu_1^{(M)} - xI_1^t - \textstyle\sum_{j=1}^{M}(-1)^j\{\sigma(u_j^{(M)}) + V_1^tu_{j+1}^{(M)}\}I_1^t\cdots I_j^t \\ &\qquad + (-1)^M I_1^t\cdots I_M^t I_{M+1}^t\Big\} \\ &= (-1)^{M+1}I_1^t x. \quad (\because \text{Lemma A.2 and A.3}) \quad ■ \end{aligned}$$

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